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On the application of Liao's method for solving linear systems

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Abstract In this paper, an analytical attitude is proposed for solving linear systems by Homotopy Analysis Method (HAM). On the basis of HAM we design new iterative methods. The convergence properties of the proposed method have been analyzed. Numerical examples show that our method is effective and simple for applications.

1. Introduction

Consider the following system of linear equation

$$AX = b, \quad (1)$$

where, X denotes a vector in a finite-dimensional space and $A \in R^{n\times n}$.

Many problems that arise in technology, industry and science are linear systems. There are some reliable methods for solving this class of problems ([1–4] and the references therein). Here, we use alternative approach to solve (1) based on the homotopy analysis method (HAM). The homotopy analysis method (HAM), first proposed by Liao in 1992 and was further developed and improved by him ([5–8] and the references therein). Liao, presented a homotopy analysis technique based on the introduction of homotopy in topology coupled with the traditional perturbation method for the solution of nonlinear problems, but unlike the traditional perturbation methods, Liao's method does not require a small perturbation parameter in the equation. In this method, a homotopy with an imbedding parameter is constructed, and the imbedding parameter is considered as a small parameter. Thus the original problem is converted into an infinite number of linear problems without using the perturbation techniques.

In recent years, this method has been successfully employed to solve many types of different problems in science and engineering. Ayub et al. [9] studied the problem of steady flow of a third grade fluid past an infinite porous plate and by using HAM, obtained an exact analytical solution of the governing non-linear differential equation. Wang and Pop [10] based on HAM, proposed exact analytic solutions for flow within a non-Newtonian fluid film whose motion is caused solely by the unsteady stretching of a horizontal elastic surface. Wang et al. [11] extended the HAM to consider the explicit analytic solution of the Volterra equation. Liang and Jeffrey [12] compared HAM and another homotopy method, called the

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homotopy perturbation method (HPM) through a linear partial differential equation. Abbasbandy and Shirzadi [13] based on HAM, obtained numerically approximate the eigenvalues of the fractional Sturm–Liouvile problems. Nassar et al. [14] surveyed that the application of the HAM can be used to obtain extremely good analytical approximations to the solution of the nonlinear Poisson–Boltzmann equation for semiconductor devices. All of these successful applications verified the validity, effectiveness and flexibility of the HAM. In this paper we focus on the use of the homotopy analysis method for solving the linear systems and conditions are deduced to check the convergence of the HAM series.

2. Homotopy analysis method for linear system

A homotopy between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H{:}X\times[0,1]\to Y$, from the product of the space X with the unit interval $[0,1]$ to Y such that, for all points $x\in X$, $H(x,0)=f(x)$ and $H(x,1)=g(x)$. If we think of the second parameter of H as "time", then H describes a "*continuous deformation*" of f into g. At time $t=0$, we have the function f and at time $t=1$ we have the function g.

In this section we propose a new numerical algorithm for solving linear systems based on homotopy analysis method (HAM).

To illustrate this method, we consider the linear Eq. (1), where;

$$A=[a_{ij}],\quad b=[b_i],\quad x=[x_j],\quad i=1,2,\dots,n\quad j=1,2,\dots,n.$$

Let $N(y)=Ay-b$ and $L(y)=My$. Then, the Liao's [6] *zero-order deformation equation*, is as follows:

$$(1-q)L[\phi(q)-y_0]=\hbar H(y)q\{A(\phi(q))-b\},\tag{2}$$

where $q\in[0,1]$ is an embedding parameter, $\hbar\neq 0$ is a non-zero auxiliary parameter, $H(y)$is an auxiliary function, L is an auxiliary linear operator, y_0 is an initial guess of y, $\varphi(q)$ is a unknown function and M is a part of splitting of A(i.e $A=M-N$) where is nonsingular, respectively. It is important to note that one has great freedom to choose the some objects such as $\hbar$, $H(y)$ and M in HAM.

Obviously, when $q=0$ and $q=1$, we will have

$$\varphi(0)=y_0,\tag{3}$$

$$\varphi(1)=y.\tag{4}$$

So, as q increases from 0 to 1, $\varphi(q)$ varies (or deforms) from the initial guess y_0 to the solution y. Expanding $\varphi(q)$ in Taylor series with respect to q, we have

$$\phi(q)=y_0+\sum_{k=1}^{\infty}y_kq^k,\tag{5}$$

where

$$y_k=\left(\frac{1}{k!}\frac{\partial^k\phi(q)}{\partial q^k}\right)_{q=0}.\tag{6}$$

If L, y_0, $\hbar$, $H(y)$ and M properly chosen, then the above series converges when $q=1$ and

$$y=y_0+\sum_{k=1}^{\infty}y_k,\tag{7}$$

must be one solution of linear system (1) as proved by Liao [6]. It should be emphasized that it is very important to ensure that the series (5) converges for $q=1$. Otherwise, the series (7) has no meaning. As $\hbar=-1$ and $H(y)=I$, Eq. (2) becomes

$$(1-q)L[\phi(q)-y_0]+q\{A(\phi(q))-b\}=0,\tag{8}$$

which is mostly been used in the homotopy perturbation method proposed in 1998 by He [15–19].

According to (6), the governing equations can be deduced from the *zeroth-order deformation* Eq. (2).

Define the vectors

$$\vec{y}(n)=\{y_0,y_1,\dots y_n\}.\tag{9}$$

Differentiating (2), k times with respect to the embedding parameter q and then setting $q=0$ and finally dividing them by $k!$, we have the *kth-order deformation equation*:

$$L[y_k-\chi_k y_{k-1}]=\hbar H(y)\frac{1}{(k-1)!}\left.\frac{\partial^{k-1}[A(\phi(q))-b]}{\partial q^{k-1}}\right|_{q=0}=0.\tag{10}$$

where,

$$\chi_k=\begin{cases}0, & k\leqslant 1,\\ 1, & else.\end{cases}$$

The right-hand side of (10) will depend only on the terms y_r with $r<k$. As a result, the terms y_k can be obtained in order of increasing k by solving the linear deformation equations in succession. The solution to the *kth-order deformation equation* can be written as

$$y_k=y^h+y_k^t,\tag{11}$$

where y^h satisfies the homogeneous equation

$$L[y^h]=0,\tag{12}$$

Here, by definition of L we can choose $y^h=0$.

And y_k^t is a particular solution of (10). We can express y_k^t as

$$y_k^t=\chi_k y_{k-1}+M^{-1}\left(\hbar H(y)\frac{1}{(k-1)!}\left.\frac{\partial^{k-1}[A(\phi(q))-b]}{\partial q^{k-1}}\right|_{q=0}\right),\tag{13}$$

where M^{-1} is the inverse operator of the M. We now define the *mth partial sum of the terms* y_k as

$$y^m=\sum_{p=1}^{m}y_p.\tag{14}$$

The solution to (1) can then be expressed as

$$y=\phi(1)=\sum_{k=1}^{\infty}y_k=\lim_{m\to\infty}y^m.\tag{15}$$

This solution will be valid wherever the series converges.

With these definitions and Eq. (7), the right hand side of (10) becomes

$$\begin{aligned}\hbar H(y)\frac{1}{(k-1)!}\left.\frac{\partial^{k-1}[A(\phi(q))-b]}{\partial q^{k-1}}\right|_{q=0}
&=\hbar H(y)\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial q^{k-1}}[A(\phi(q))-b]_{q=0}\\
&=\hbar H(y)\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial q^{k-1}}\left[A\left(\sum_{k=0}^{\infty}y_k\right)-b\right]_{q=0}\\
&=\hbar H(y)[Ay_{k-1}-(1-\chi_k)b]\end{aligned}\tag{16}$$

From (11), (13) and (16) we find that

$$y_k = \chi_k y_{k-1} + M^{-1}(\hbar H(y)[Ay_{k-1} - (1-\chi_k)b]). \quad (17)$$

If we take $y_0 = 0$, then from these terms we can obtain the first several approximations to the solution of (1);

$$\begin{aligned}
y_1 &= -\hbar M^{-1}H(y)b,\\
y_2 &= y_1 + M^{-1}(\hbar H(y)Ay_1) = (I+\hbar M^{-1}H(y)A)\\
y_1 &= (I+\hbar M^{-1}H(y)A)(-\hbar M^{-1}H(y)b),\\
y_3 &= y_2 + M^{-1}(\hbar H(y)Ay_2) = (I+\hbar M^{-1}H(y)A)\\
y_2 &= (I+\hbar M^{-1}H(y)A)^2(-\hbar M^{-1}H(y)b),\\
&\vdots
\end{aligned}$$

And in general

$$y_n = (I+\hbar M^{-1}H(y)A)y_{n-1}, \quad n = 1,2,\ldots$$

If we take $H(y) = I$, then the solution can be written as

$$y = \sum_{i=0}^{\infty}(I+\hbar M^{-1}A)^i(-\hbar M^{-1}b). \quad (18)$$

For the convergence of the above method we refer the reader to Liao's work [6]. Furthermore, the following Lemma is useful for the convergence of above method.

Lemma 2.1 20. *The series $\sum_{i=0}^{\infty}A^i$ convergence if and only if the spectral radius of A less than one (i.e, $\rho(A)<1$).*

Therefore the series (18) converges if and only if $\rho(I + \hbar M^{-1}A) < 1$.

We can prove that the convergence of the above method for a class of well-known matrices as the following;

Definition 2.1 (*1–4*). A real $n\times n$ matrix $A = (a_{ij})$ is called

(i) a *Z-matrix* if and only if for any $i\neq j$, $a_{ij}\leqslant 0$;
(ii) an *M-matrix* if and only if it is a *Z-matrix*, nonsingular and $A^{-1}\geqslant 0$;
(iii) an *H-matrix* if and only if $\langle A\rangle = (m_{ij})\in R^{n\times n}$ is an *M-matrix*, where

$$m_{ii} = |a_{ii}|, \quad m_{ij} = -|a_{ij}|, \quad i\neq j \quad 1\leqslant i,j\leqslant n;$$

Definition 2.2 (*1–4*). Let A be a real matrix. The splitting $A = M - N$,

(i) Converges if $\rho(M^{-1}N) < 1$.
(ii) Is called *M-splitting* if M is a nonsingular *M-matrix* and $N\geqslant 0$.

Lemma 2.2 (*1–4*). *Let $A = M - N$, be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular M-matrix.*

Lemma 2.3 (*2,3*). *If A is an H-matrix, then $|A^{-1}| \leqslant \langle A\rangle^{-1}$.*

Theorem 2.4. *Let $A = D - L - U$, where D is the diagonal and $-L$, $-U$, are strictly lower and upper triangular matrices of A, respectively. Then the series (18) in HAM method converges if A is an H-matrix, $-1\leqslant \hbar < 0$ and $M = D$ or $M = D - L$.*

Proof. We only prove $M = D$; $M = D - L$ can be similarly verified.

For $M = D$;

$$\rho(I+\hbar D^{-1}A) = \rho(\hbar D^{-1}(\hbar^{-1}D + A)).$$

Since $\hbar < 0$,we consider $\hbar = -\alpha$ such that $0 < \alpha\leqslant 1$. Then

$$\begin{aligned}
\rho(I+\hbar D^{-1}A) &= \rho(-\alpha D^{-1}((-1/\alpha)D + A))\\
&= \rho\left(-\alpha D^{-1}\left(-\left(\frac{1-\alpha}{\alpha}\right)D - (L+U)\right)\right).
\end{aligned}$$

Since $0 < \alpha\leqslant 1$, $A = M - N$ is an *H-matrix*, we see that M is also an *H-matrix* and

$$\langle A\rangle = \underbrace{\frac{1}{\alpha}|-D|}_{M} - \underbrace{\left(\frac{1-\alpha}{\alpha}|-D| + |L| + |U|\right)}_{N},$$

is an *M-matrix* and $\langle A\rangle = M - N$ is *M-splitting*. Therefore, by Definition 2.2 and Lemma 2.2 we have

$$\rho\left(\alpha|D|^{-1}\left(\left(\frac{1-\alpha}{\alpha}\right)|D| + (|L|+|U|)\right)\right) < 1.$$

Furthermore, From Lemma 2.3, $|D^{-1}| \leqslant |D|^{-1}$ and,

$$\begin{aligned}
\left|\alpha D^{-1}\left(\frac{1-\alpha}{\alpha}D + L + U\right)\right| &\leqslant \alpha|D^{-1}|\left|\left(\frac{1-\alpha}{\alpha}D + L + U\right)\right|\\
&\leqslant \alpha|D|^{-1}\left(\frac{1-\alpha}{\alpha}|D| + |L| + |U|\right).
\end{aligned}$$

So that

$$\begin{aligned}
&\rho\left(-\alpha D^{-1}\left(-\left(\frac{1-\alpha}{\alpha}\right)D - (L+U)\right)\right)\\
&\leqslant \rho\left(\left|\alpha D^{-1}\left(\frac{1-\alpha}{\alpha}D + L + U\right)\right|\right)\\
&\leqslant \rho\left(\alpha|D|^{-1}\left(\left(\frac{1-\alpha}{\alpha}\right)|D| + (|L|+|U|)\right)\right) < 1.
\end{aligned}$$

And the proof is completed. □

3. Numerical examples

In this section, we give some examples to illustrate the results obtained in the previous section

Example 3.1. The coefficient matrix A of (1) is given by:

$$A = \begin{bmatrix} 5 & 1 & 0 & 2 & -1\\ 3 & 6 & 0 & -1 & 1\\ 2 & -1 & 7 & -2 & 1\\ -3 & 0 & 1 & 8 & 2\\ 1 & 5 & 1 & 2 & 9 \end{bmatrix},$$

And $b = [17 \quad 27 \quad 39 \quad 76 \quad 127]^t$. The exact solution is $X = [2 \quad 3 \quad 6 \quad 7 \quad 10]^T$.

Matrix A is an *H-matrix*. From (18), when $\hbar = -1$ and $H(y) = I$, by using some terms we have,

$$y_0 = \begin{bmatrix} 3.4000 \\ 4.5000 \\ 5.5714 \\ 9.5000 \\ 14.1111 \end{bmatrix}, \quad y_1 = \begin{bmatrix} -1.8778 \\ -2.4685 \\ 0.3698 \\ -2.9492 \\ -5.60791 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 0.5518 \\ 1.3820 \\ 0.1424 \\ 0.6516 \\ 2.1943 \end{bmatrix},$$

$$y_3 = \begin{bmatrix} -0.0982 \\ -0.5330 \\ -0.0875 \\ -0.3595 \\ -0.9897 \end{bmatrix},$$

$$y_4 = \begin{bmatrix} 0.0524 \\ 0.1541 \\ -0.0094 \\ 0.2216 \\ 0.3966 \end{bmatrix}, \quad y_5 = \begin{bmatrix} -0.0401 \\ -0.0554 \\ 0.0137 \\ -0.0783 \\ -0.13961 \end{bmatrix}, \quad y_6 = \begin{bmatrix} 0.0145 \\ 0.0303 \\ 0.0011 \\ 0.0182 \\ 0.0511 \end{bmatrix},$$

$$y_7 = \begin{bmatrix} -0.0031 \\ -0.0127 \\ -0.0019 \\ -0.0075 \\ -0.0226 \end{bmatrix},$$

$$y \approx \sum_{i=0}^{7} y_i = \begin{bmatrix} 1.9996 \\ 2.9967 \\ 5.9996 \\ 6.9968 \\ 9.9933 \end{bmatrix}.$$

Example 3.2. Consider the following convection-diffusion equation

$$-\underbrace{(u_{xx} + u_{yy})}_{\Delta u} + \delta u_x + \tau u_y = f(x, y),$$

on the unit square domain $\Omega = [0,1] \times [0,1]$, with constant coefficients δ, τ and subject to Dirichlet boundary conditions. Discretization by a five-point finite difference operator leads to a linear system of (1). Where X denotes a vector in a finite-dimensional space and $A \in R^{n^2 \times n^2}$. With discretization on a uniform $n \times n$ grid, using standard second-order differences for the Laplacian, and either centered or upwind differences for the first derivatives.

The coefficient matrix has the form

$$A = tridiagonal[bI, tridiagonal[c, a, d], eI],$$
$$b = -(1 + \partial); c = -(1 + \gamma); a = 4; d = -(1 - \gamma); e = -(1 - \partial).$$

Or with Kronecker product ($\otimes$) we obtain,

$$A = I \otimes T_x + I \otimes T_y, \quad T_x = tridiagonal[c, a, d], \quad T_y = tridiagonal[bI, 0, eI],$$

where $\partial = \frac{\tau h}{2}; \gamma = \frac{\delta h}{2}$ are Reynolds numbers. Furthermore, the equidistant step-size $h = 1/n$ is used in the discretization and the natural lexicographic ordering is employed to the unknowns and the right-hand side satisfies $b_{ij} = h^2 f_{ij}(x, y)$. For details, we refer to ([21,22] and the references therein).

Table 1 Shows the results of Example 3.2.

n	N	ω	ħ	Iter	CPU
10	100	0.0	−0.5	147	1.453311
			−0.7	99	1.055267
			−1.0	63	0.661414
			−1.1	77	0.852654
10	100	0.8	−0.5	112	1.203719
			−0.8	62	0.676677
			−1.1	38	0.428743
			−1.3	70	0.715836
10	100	1.1	−0.5	96	1.105107
			−0.9	41	0.520153
			−1.1	28	0.170774
			−1.3	45	0.608243
15	225	0.0	−0.5	184	22.663019
			−0.8	107	11.570875
			−1.0	81	7.940693
			−1.1	127	15.044778
15	225	0.8	−0.5	141	16.330652
			−0.9	68	6.807409
			−1.1	50	4.271111
			−1.2	64	5.950185
15	225	1.1	−0.5	123	13.695164
			−1.0	45	4.526209
			−1.1	37	3.419221
			−1.3	54	5.154205

We test convection-diffusion equation, when $\delta = 1$, $\tau = 2$. Then, we solved the $n^2 \times n^2$ *M-matrix* yielded by HAM. In this experiment, we choose the right hand side vector, such that $X = (1, 1, \ldots, 1)^T$ be solution of $AX = b$. As a stopping criterion we choose $\left(\left\|X - \sum_{i=0}^{Iter} y_i\right\|_2 / \|X\|_2\right) \leqslant 10^{-10}$.

All the numerical experiments presented in this section were computed in double precision using a MATLAB7 on a PC with a 1.86 GHz 32-bit processor and 1 GB memory.

In Table 1 we report the CPU time and number of iterations (Iter) HAM method for $M = D - \omega L$ with different ω, ħ and $N = n^2$.

4. Conclusions

In this paper, the homotopy analysis methods (HAM) is applied for solving the system of Linear equations and derive conditions to check the convergence of this analytical method. Finally, some numerical examples have been provided to illustrate that the present method is successful in accuracy and convergence speed. Our method yields very accurate approximate solutions using only few iterates.

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